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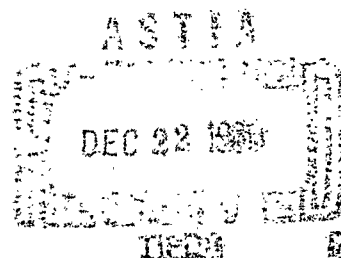
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CURRENT AND MAGNETIC FIELD DISTRIBUTION  
FOR AN INFINITELY LONG SUPERCONDUCTOR  
OF RECTANGULAR CROSS-SECTION

G. J. Culler



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**G. J. Culler**

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G. P. Celler

I. INTRODUCTION

The numerical results contained in this report give an essentially complete picture of the quantitative manner in which current piles up along the corners of a superconductor. We have chosen various cross-sections varying down to dimensions comparable to the skin depth as we will later wish to compare the results obtained here (using the local London equation) with those to be obtained from the non-local Pippard equation.

## II. TWO REGION PROBLEM

We assume that inside the superconductor current density and magnetic field are related by the equations

$$\begin{aligned}\nabla \times (\lambda^2 \vec{j}) &= \vec{H} && \text{(London)} \\ \nabla \times \vec{H} &= \vec{j} && \text{(Maxwell)}\end{aligned}\tag{II.1}$$

It is convenient to choose a rectangular coordinate system with the Z-axis coinciding with the longitudinal axis of the superconductor. Let  $\vec{A}$  be a vector potential such that  $\vec{H} = \nabla \times \vec{A}$  with the gauge  $\nabla \cdot \vec{A} = 0$ , then the only non-vanishing component of  $A$  is  $A_z$  which we will write as the scalar function  $\mathcal{J}(x, y)$ . The current density and the magnetic field are given in terms of  $\mathcal{J}$ ,

$$H_x = \frac{\partial \mathcal{J}}{\partial y}, \quad H_y = -\frac{\partial \mathcal{J}}{\partial x}, \quad j_z = -\frac{1}{\lambda^2} \mathcal{J}\tag{II.2}$$

Using the Maxwell equation outside the superconductor and combining Eqs. (II.1), we have

$$\nabla^2 \mathcal{J} = \begin{cases} 0 & \text{outside} \\ \frac{1}{\lambda^2} \mathcal{J} & \text{inside,} \end{cases}\tag{II.3}$$

where the field outside is still given by (II.2). At the superconductor boundary  $H_x$  and  $H_y$  must be continuous; this condition is fulfilled with no

loss of generality if  $\phi$  and its normal derivative are required to be continuous across the boundary.

Since (II.3) is homogeneous, the normalization of  $\phi$  may be chosen at will. Now far away from the conductor,  $|\vec{H}| = \frac{I}{2\pi r}$ , so from (II.2) the total current is

$$I = 2\pi \lim_{r \rightarrow \infty} r \left| \frac{d\phi}{dr} \right| \quad (\text{II.4})$$

Our mathematical and computational problem is then to solve the two-regional problem (II.3) by a function which is continuous and has continuous first derivatives everywhere and which is asymptotically logarithmic.



### III. REGIONAL ITERATION

The iterative method selected for solving (II.3) consists of alternately solving an exterior and interior problem. Let  $C$  denote the boundary of the rectangle and suppose  $\phi_0(c)$  is an initial guess for the boundary values of  $\phi$ . Let  $\phi^i$  and  $\phi^e$  denote solutions of the interior and exterior problems respectively.

The iteration is started by solving

$$\nabla^2 \phi_1^e = 0 \quad , \quad \phi_1^e(c) = \phi_0(c) \quad (\text{III.1})$$

and continued by solving

$$\begin{aligned} \nabla^2 \phi_k^i &= \frac{1}{\lambda^2} \phi_k^i \quad , \quad \frac{\partial \phi_k^i(c)}{\partial n} = \frac{\partial \phi_k^e(c)}{\partial n} \\ \nabla^2 \phi_{k+1}^e &= 0 \quad , \quad \phi_{k+1}^e(c) = \phi_k^i(c) \end{aligned} \quad (\text{III.2})$$

Since we must ultimately match both the values of the functions and their normal derivatives, there are many ways to choose the boundary conditions other than those selected above. To see that this choice is not arbitrary, consider a one-dimensional analogue of (II.3),

$$\begin{aligned} \frac{d^2 \phi}{dx^2} &= \frac{1}{\lambda^2} \phi \quad , \quad |x| < L \\ \frac{d^2 \phi}{dx^2} &= 0 \quad , \quad |x| > L \end{aligned} \quad (\text{III.3})$$

To effect a normalization of  $\varphi$ , set  $\varphi(KL) = 1$ ,  $K > 1$ , then

$$\varphi(x) = A \cosh\left(\frac{x}{\lambda}\right), \quad \varphi(x) = 1 + B(x - KL) \quad (\text{III.4})$$

According to our proposed iterative method, we compute

$$\frac{1}{\lambda} A_n \sinh\left(\frac{L}{\lambda}\right) = B_n \quad (\text{III.5})$$

$$1 + B_{n+1}(1-K)L = A_n \cosh\left(\frac{L}{\lambda}\right)$$

that is,

$$B_{n+1} = \frac{1}{L(K-1)} - \frac{\lambda \coth\left(\frac{L}{\lambda}\right)}{L(K-1)} B_n \quad (\text{III.6})$$

and the condition for convergence is

$$\frac{\lambda}{L} \frac{\coth\left(\frac{L}{\lambda}\right)}{K-1} < 1 \quad (\text{III.7})$$

In our problem, we are principally interested in large  $K$  so this is the appropriate order of the iteration. Reversing the above procedure leads to the reciprocal of the above condition; this would be the right choice for  $K$  near 1.

#### IV. EXTERIOR PROBLEM

Since  $\mathcal{J}$  is known to be constant on sufficiently large circles, one could conceivably specify  $\mathcal{J}$  on such a circle and attempt to solve the exterior problem by a relaxation method for some difference approximation of Laplace's equation. There are two substantial drawbacks to this approach; a) because of the accuracy required near the rectangular boundary an enormous number of grid points would be needed, b) there is no satisfactory way to determine how large the bounding circle should be (it turns out that it would have to be exceedingly large for long, thin rectangles). With the help of the conformal mapping developed below, we are able to reduce the problem to series form and thereby treat the whole infinite region.

Consider a quadrant of the exterior as shown below.

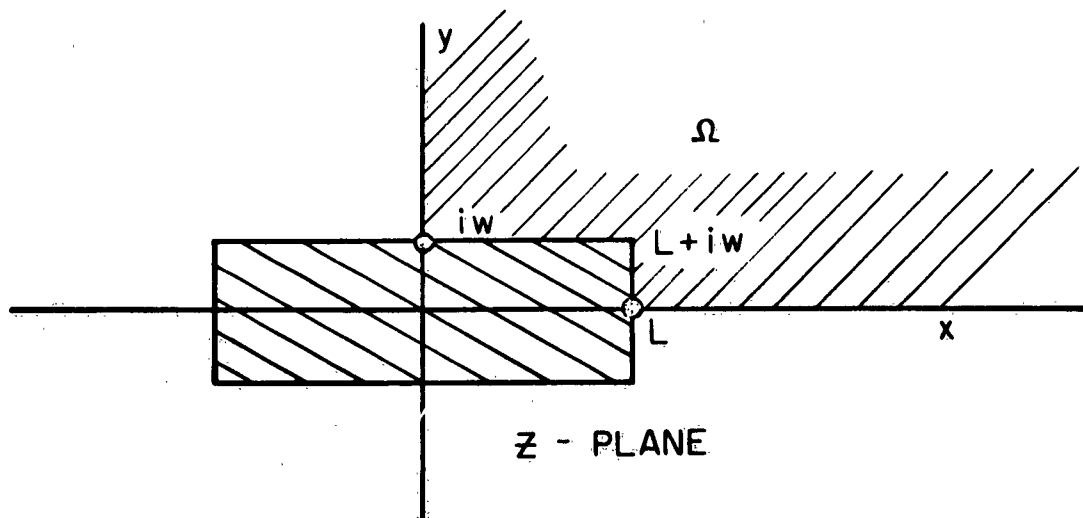


Figure IV.1 Quadrant of the Region Exterior to the Rectangle

The region  $\Omega$  is the conformal image of a half-strip in the  $w$ -plane under the mapping (c.f. Appendix A),

$$Z(w) = N(iB(\alpha) + \int_0^w \sqrt{\cos \tau - \cos \alpha} d\tau) \quad (\text{IV.2})$$

Where  $W = NB(\alpha)$ ,  $L = NA(\alpha)$  and

$$A(\alpha) = \int_0^\alpha \sqrt{\cos \tau - \cos \alpha} d\tau, \quad B(\alpha) = \int_\alpha^\pi \sqrt{\cos \alpha - \cos \tau} d\tau. \quad (\text{IV.3})$$

On this half-strip, the solution of Laplace's equation is given by the series

$$\Phi(u, v) = v + \sum_{n=0}^{\infty} a_n \cos(nu) e^{-nv} \quad (\text{IV.4})$$

and hence  $\phi(x, y) = \Phi(u(x, y), v(x, y))$  is obtained by inverting (IV.2).

The coefficient of  $v$  in (IV.4) determines the normalization of the asymptotic form of  $r \frac{\partial \phi}{\partial r}$ . On  $x = 0$ , we have

$$y \sim N(c + \sqrt{2} e^{-\frac{x}{2}}); \quad \frac{\partial \Phi}{\partial v} \sim 1; \quad \frac{\partial \phi}{\partial y} \sim \frac{2}{y} \quad (\text{IV.5})$$

and hence  $I = 4\pi$  is the total current for all cases computed.

## V. INTERIOR PROBLEM

The solution for the interior problem was approximated by solving a difference equation using a relaxative method. A rectangular grid was selected with a fine subdivision near the material boundary and with a comparatively coarse subdivision near the middle of the superconductor. Error estimates were obtained for this interior problem both from known local formulas and by computing a test problem of known solution.

## VI. COMPUTATION

These computations were performed on the Space Technology Laboratories' I. B. M. 7090 and required some five to ten minute computation time per case. In practice, it was found that only a few regional iterations were needed and that convergence was hastened by averaging the coefficients in successive exterior series (IV. 4). For a typical case, several thousand grid points were used in the interior region and several hundred terms were used in the exterior series. At this time it appears to the author that a series approach to the interior problem would have significant advantage over the relaxation method used.

Results of the computation are exhibited in the graphs of Appendix B. For large regions, the graphs only show the character of  $\mathcal{P}$  near a corner. The labeling  $\mathcal{P}_c$ ,  $\mathcal{P}_n$ ,  $\mathcal{P}_e$  refer to the values  $\mathcal{P}(L+iW)$ ,  $\mathcal{P}(iW)$ ,  $\mathcal{P}(L)$  as shown in Figure (IV. 1).

# APPENDIX A

The mapping used in the exterior problem.

Let  $Z(W)$  be the mapping function for the regions shown below.

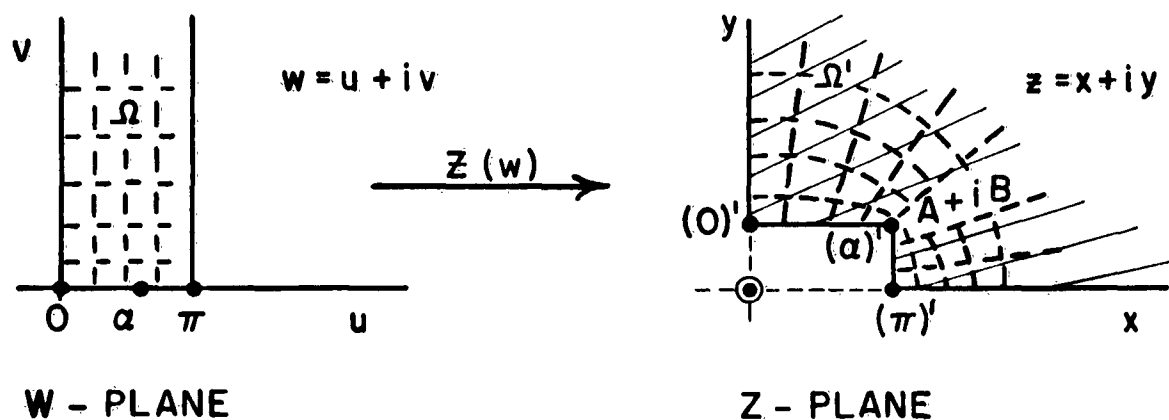


Figure A.1 Regions Defining  $Z(W)$

We will show that

$$Z(w) = iB + \int_0^w \sqrt{\cos \zeta - \cos \alpha} d\zeta$$

$$A = \int_0^\alpha \sqrt{\cos \zeta - \cos \alpha} d\zeta \tag{A.2}$$

$$B = \int_\alpha^\pi \sqrt{\cos \alpha - \cos \zeta} d\zeta$$

The fundamental net for the region exterior to a rectangle consists of curves  $U(x, y) = \text{constant}$ ,  $v(x, y) = \text{constant}$ , where  $u$  and  $v$  are conjugate

harmonic functions on this region. Cutting this region along the lines of symmetry results in the region  $\Omega'$  whose boundary is entirely made up by lines of the fundamental net, thus the mapping  $W(Z) = u(x, y) + iv(x, y)$  will map  $\Omega'$  on to a region  $\Omega$  with a rectangular fundamental net and we can determine the image  $\Omega$  of  $\Omega'$  under  $W(Z)$  as well as  $\Omega''$  under  $\frac{dW}{dz} = u_x + iv_x = v_y - iu_y$ . If we can then map  $\Omega$  onto  $\Omega''$  by a mapping  $F(W)$ , we will obtain a differential equation

$$\frac{dw}{dz} = F(w) \quad , \quad \frac{dz}{dw} = \frac{1}{F(w)} \quad (A.3)$$

which can be integrated for the desired mapping  $Z(W)$ .

The region  $\Omega''$  is constructed by considering the values of  $u_x, v_x, u_y, v_y$  on the various pieces of the boundary of  $\Omega$ .

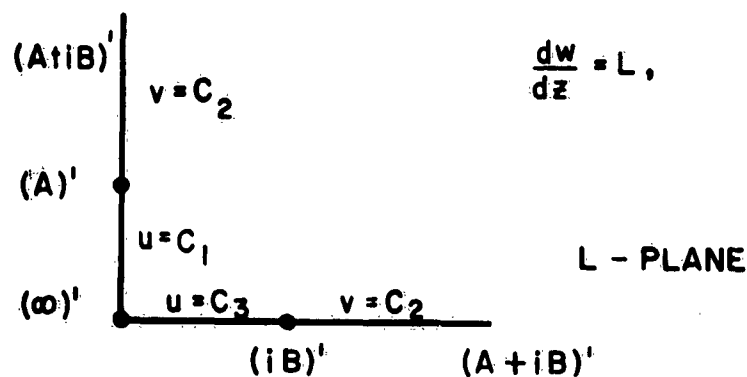


Figure A.4 The Image of  $\Omega'$  under  $\frac{dw}{dz}$ .



The image  $\Omega'''$  of  $\Omega''$  under the mapping  $1/L^2$  is:

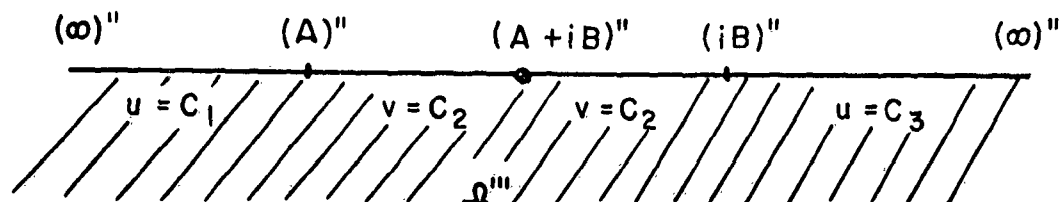


Figure A.5 An Intermediate Image

Now choose  $C_1 = \pi$ ,  $C_2 = 0$ ,  $C_3 = 0$ , and map  $\Omega'$  by  $\cos(W)$  on to  $\Omega^\circ$ , this gives

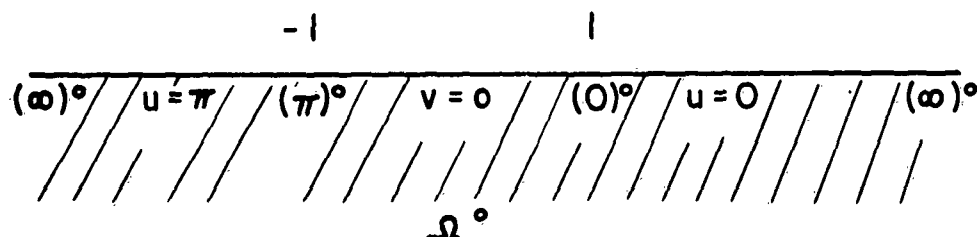


Figure A.6 An Intermediate Image

which except for translation and normalization agrees with  $\Omega'''$ . The normalization is fixed by  $(iB)'' - (A)'' = 2$  and since  $(A + iB)''$  lies between  $(A)''$  and  $(iB)''$  we can choose the translation to be  $\cos \alpha$  for  $0 < \alpha < \pi$ , then

$$\frac{1}{L^2} = \cos w - \cos \alpha \quad (\text{A.7})$$

But  $1/L = \frac{dz}{dw}$ , so  $Z(w)$  is determined by

$$\frac{dz}{dw} = \sqrt{\cos w - \cos \alpha} \quad (\text{A.8})$$

This can be developed in an exponential series by using the generating function for Legendre polynomials

$$\frac{1}{\sqrt{1-2h\cos\alpha+h^2}} = \sum_{n=0}^{\infty} h^n P_n(\cos\alpha) \quad (\text{A.9})$$

To do this, write

$$\frac{dz}{dw} = \frac{1}{\sqrt{2}} e^{-\frac{iw}{2}} \sqrt{1 - 2\cos\alpha e^{iw} + e^{2iw}} \quad (\text{A.10})$$

then

$$\begin{aligned} \frac{d}{dw} \left( \frac{1}{\sqrt{2}} e^{\frac{iw}{2}} \frac{dz}{dw} \right) &= i \frac{e^{2iw} - \cos\alpha e^{iw}}{\sqrt{1 - 2\cos\alpha e^{iw} + e^{2iw}}} \\ &= i (e^{2iw} - \cos\alpha e^{iw}) \sum_{n=0}^{\infty} e^{inw} P_n(\cos\alpha) \\ &= i \left\{ -\cos\alpha e^{iw} + \sum_{n=2}^{\infty} (P_{n-2}(\cos\alpha) - \cos\alpha P_{n-1}(\cos\alpha)) e^{inw} \right\} \end{aligned} \quad (\text{A.11})$$

This gives

$$\frac{dz}{dw} = \frac{c}{\sqrt{2}} e^{-i\frac{w}{2}} - \frac{\cos \alpha}{\sqrt{2}} e^{i\frac{w}{2}} + \frac{1}{\sqrt{2}} \sum_{n=2}^{\infty} \left( \frac{P_n(\cos \alpha) - \cos \alpha P_{n-1}(\cos \alpha)}{n} \right) e^{i(n-\frac{1}{2})w} \quad (\text{A.12})$$

but for  $e^{iw}$  near zero,  $\frac{dz}{dw} \sim \frac{c}{\sqrt{2}} e^{-i\frac{w}{2}}$ , so  $C = 1$ .

$$\therefore \frac{dz}{dw} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} C_n e^{i(n-\frac{1}{2})w}$$

$$C_0 = 1, \quad C_1 = -\cos \alpha, \quad C_n = \frac{P_n(\cos \alpha) - \cos \alpha P_{n-1}(\cos \alpha)}{n} \quad (\text{A.13})$$

## APPENDIX B

The following graphs exhibit the lines of constant current density in the superconductor and lines of constant magnetic field outside the superconductor.

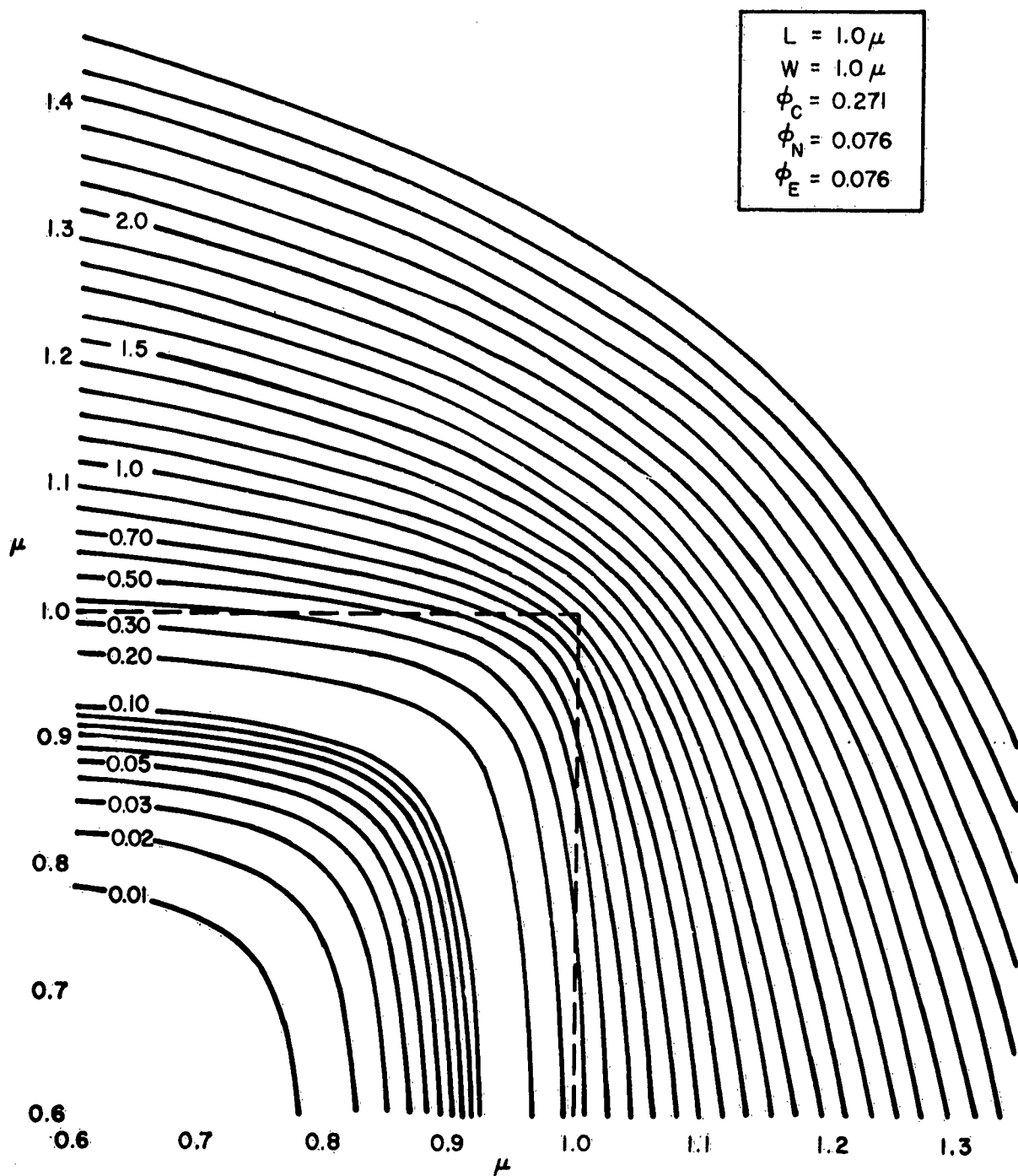


Figure B.1 - Contour Map of Relative Current Density  $\phi/\phi_c$ .

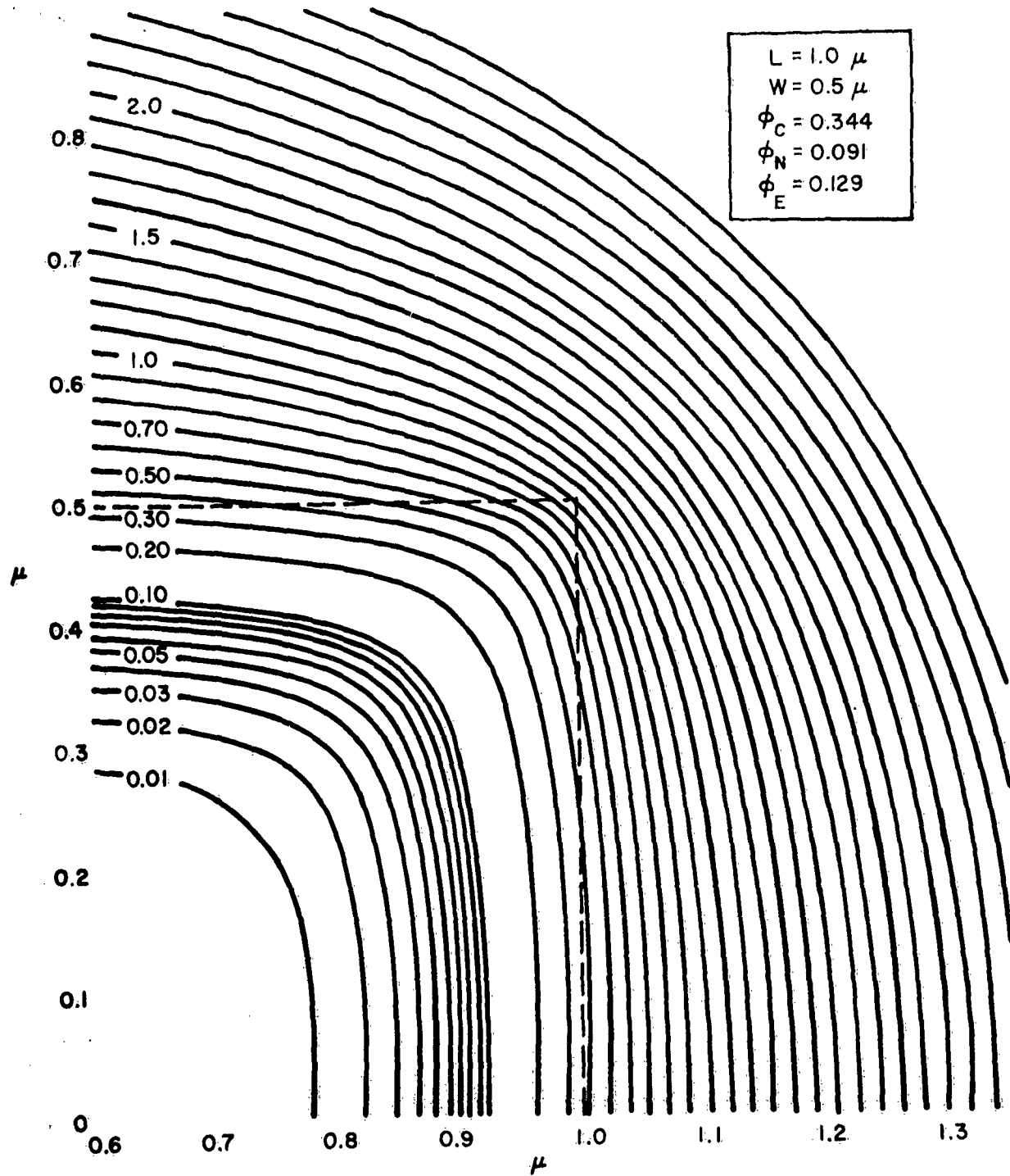


Figure B.2 - Contour Map of Relative Current Density  $J/J_c$ .

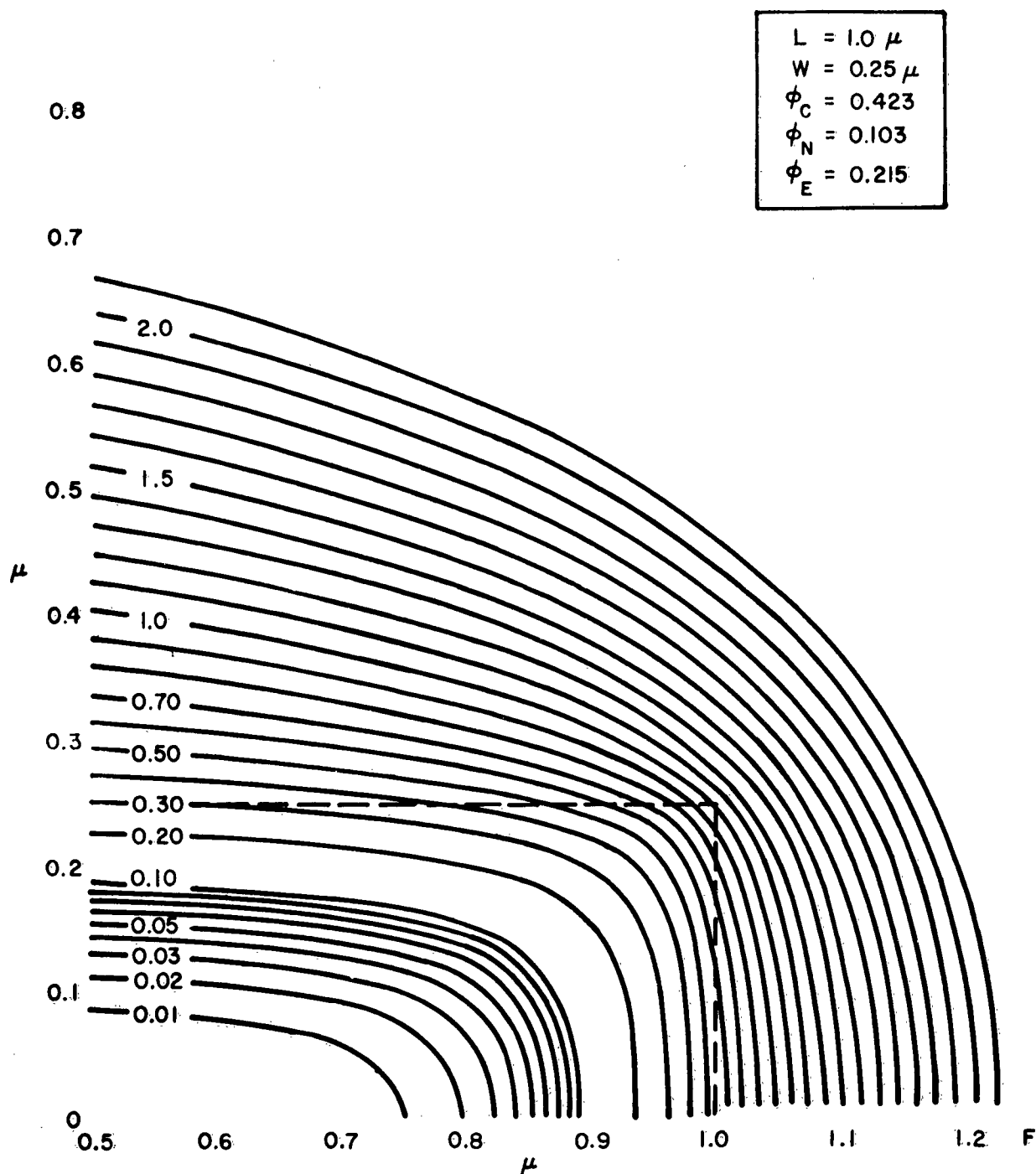


Figure B.3 - Contour Map of Relative Current Density  $\phi/\phi_C$ .

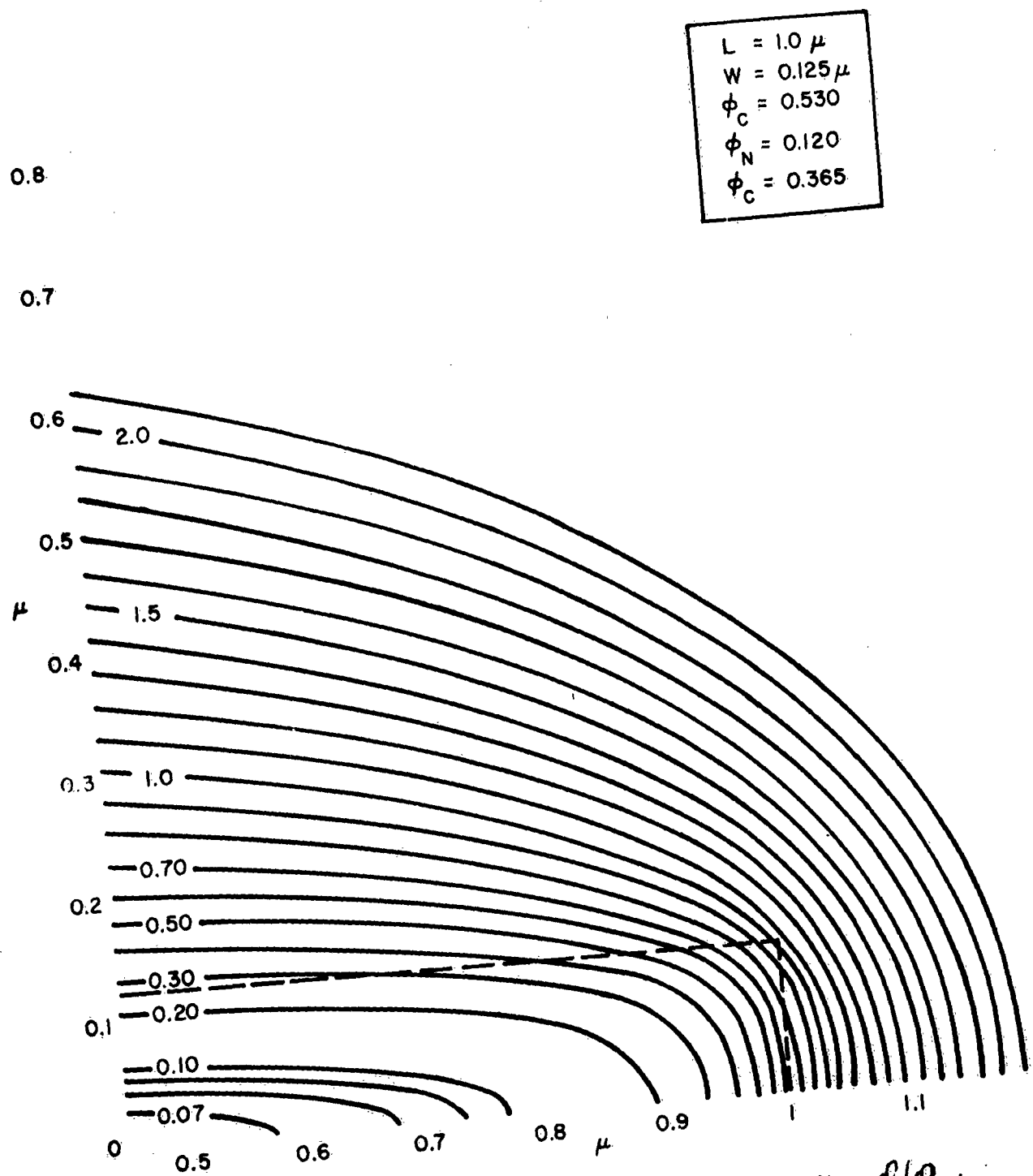


Figure B.4 - Contour Map of Relative Current Density  $g/g_c$



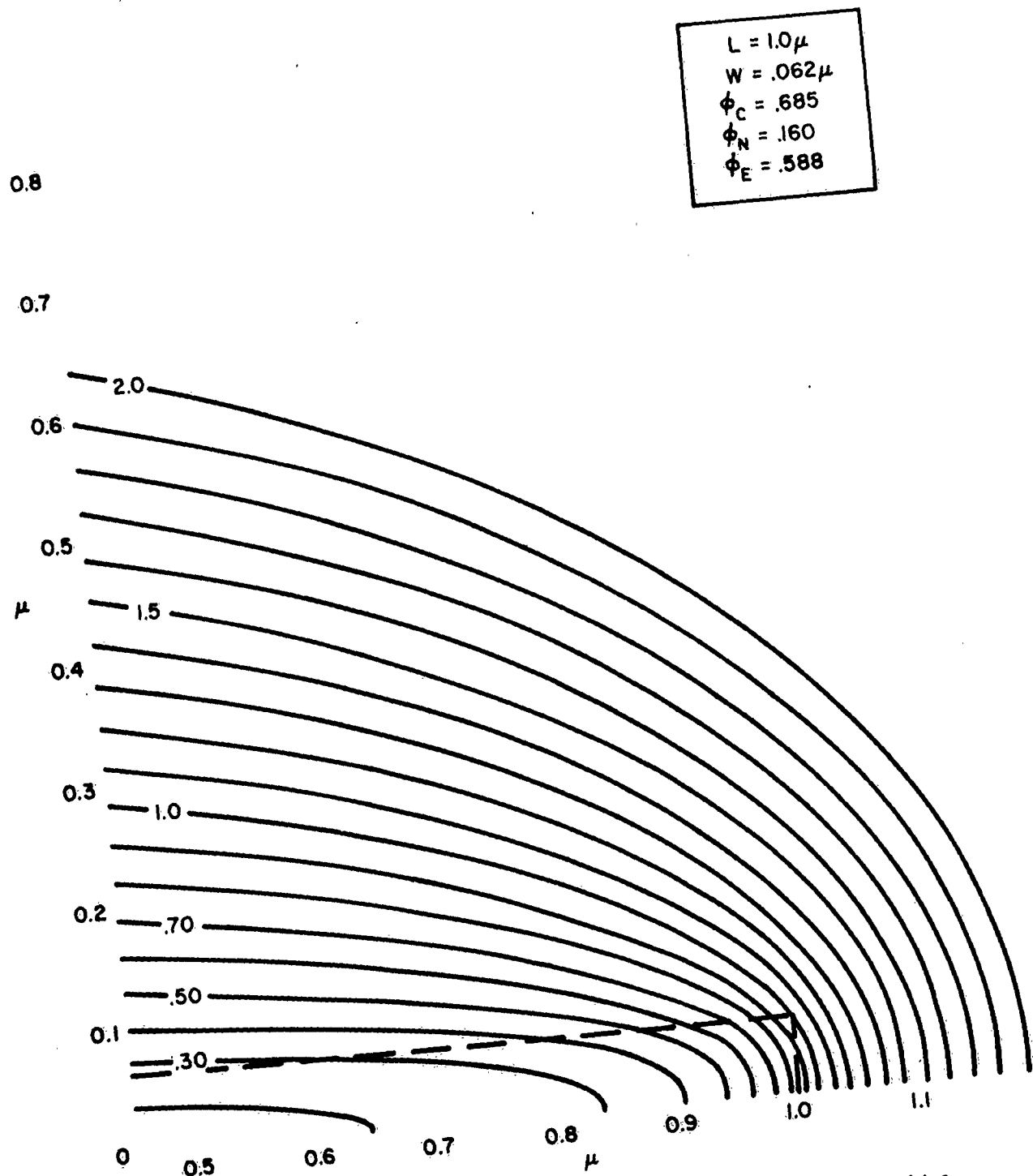


Figure B.5 - Contour Map of Relative Current Density  $J/J_c$ .

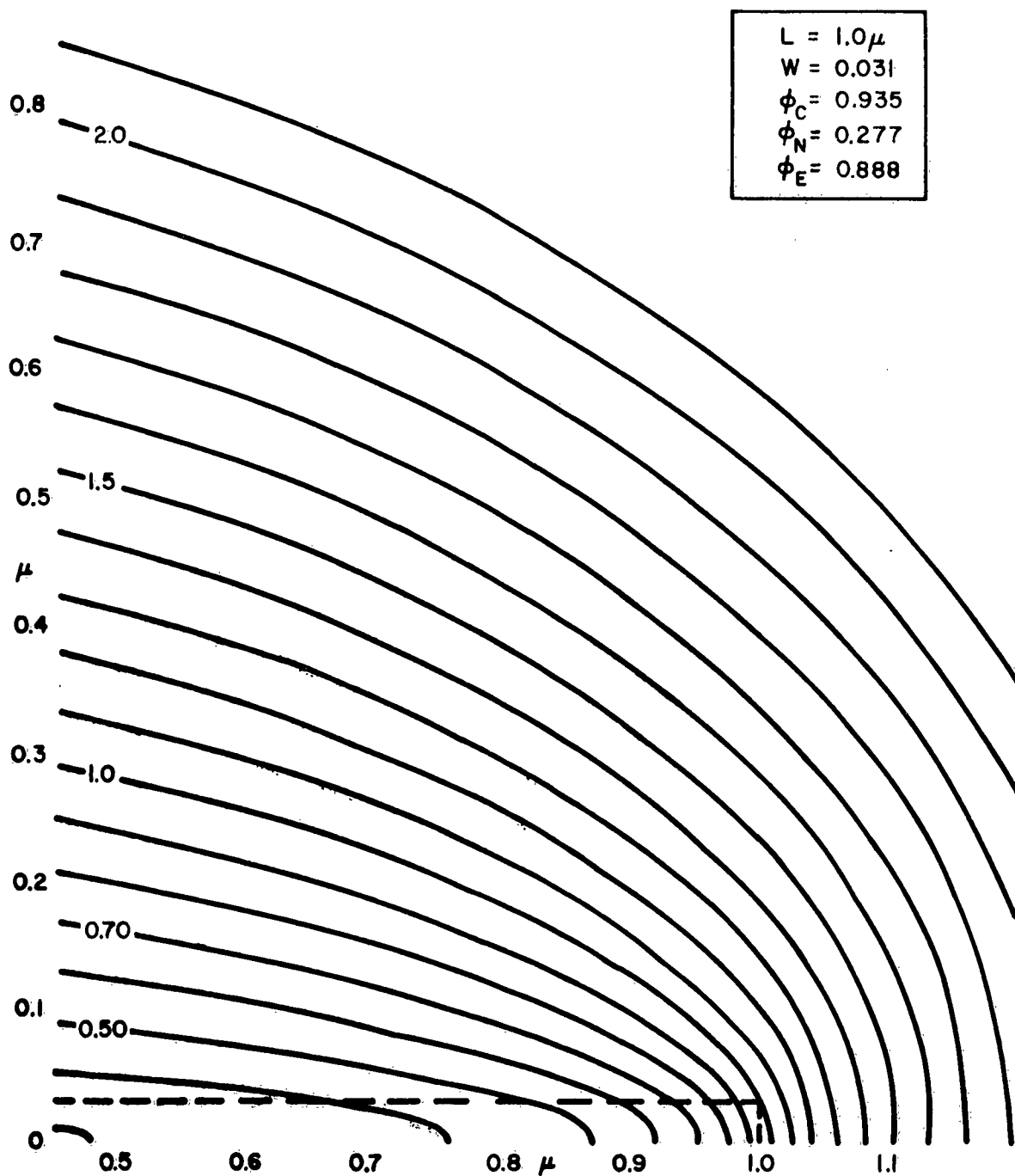


Figure B.6 - Contour Map of Relative Current Distribution  $J/J_c$ .

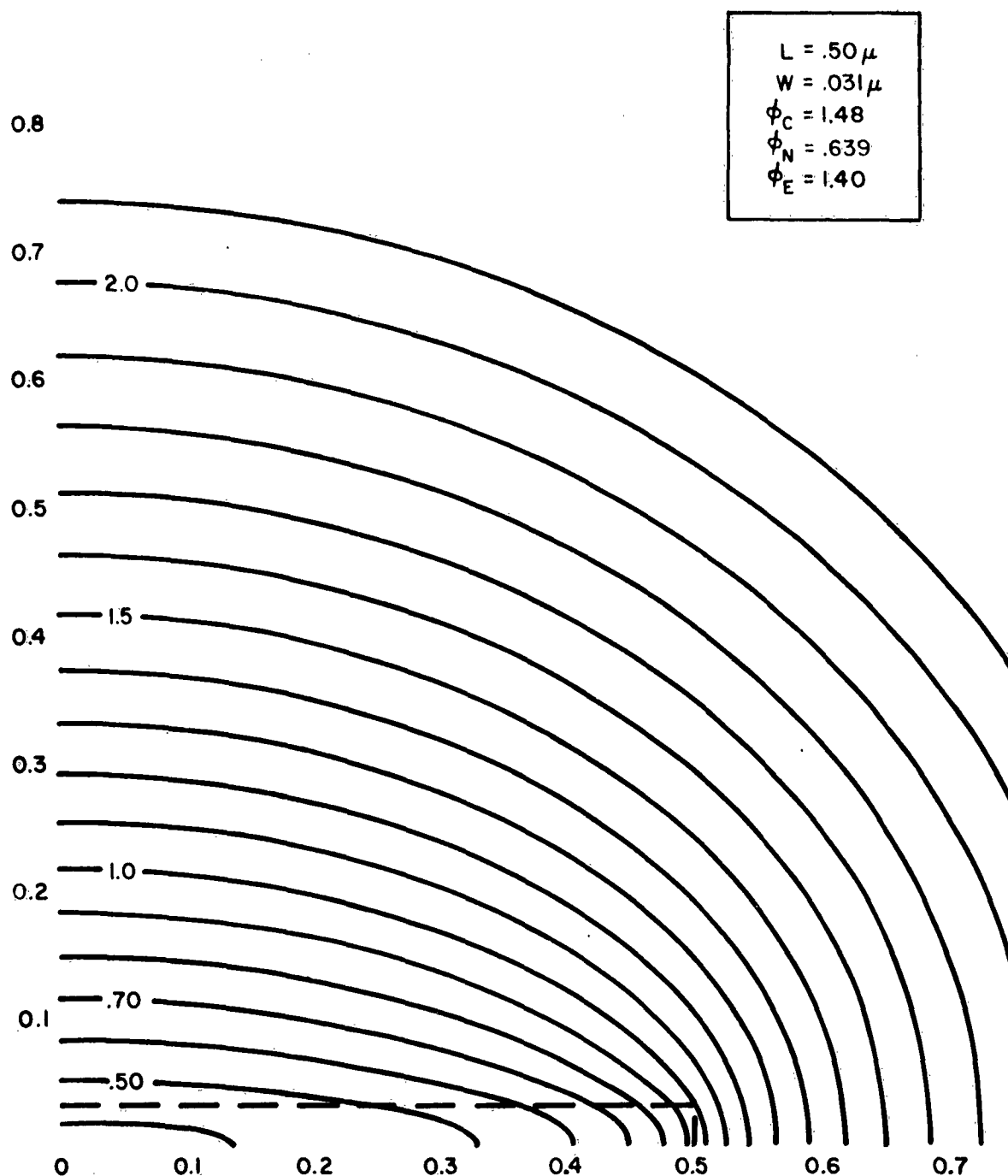


Figure B, 7 - Contour Map of Relative Current Density  $J/J_c$ .

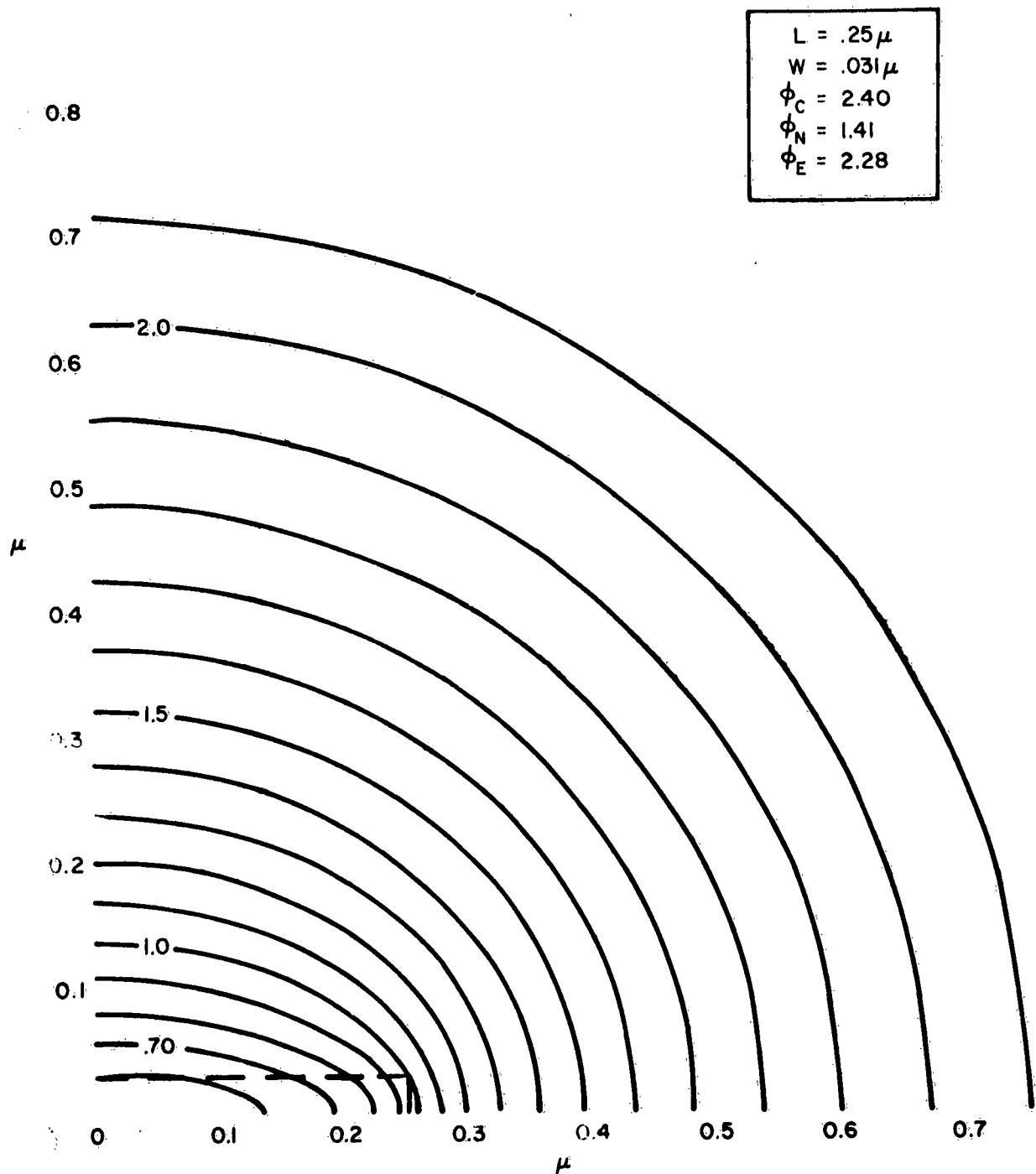


Figure B.8 - Contour Map of Relative Current Density  $J/J_c$ .

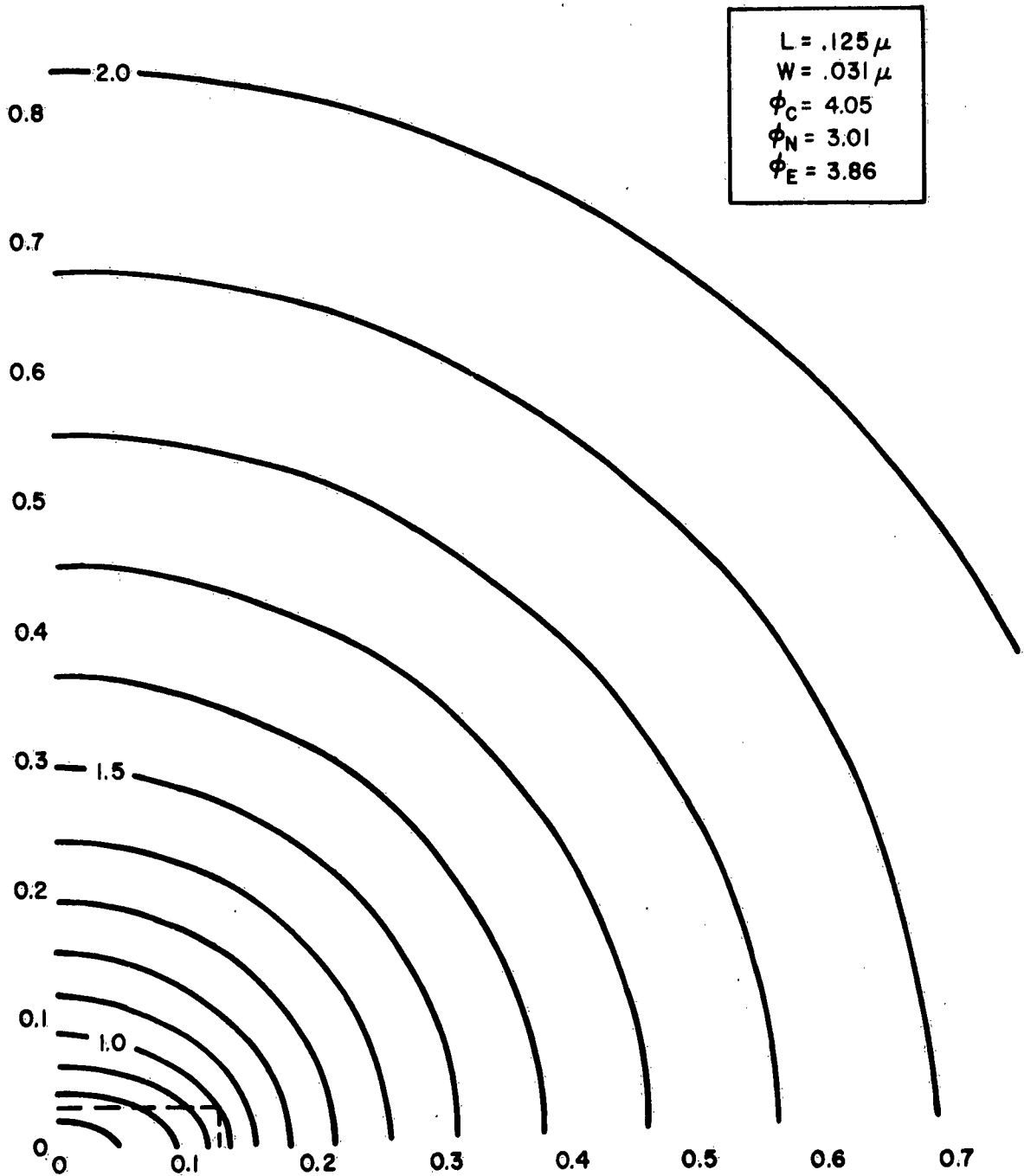


Figure B.9 - Contour Map of Relative Current Density  $\phi/\phi_c$ .

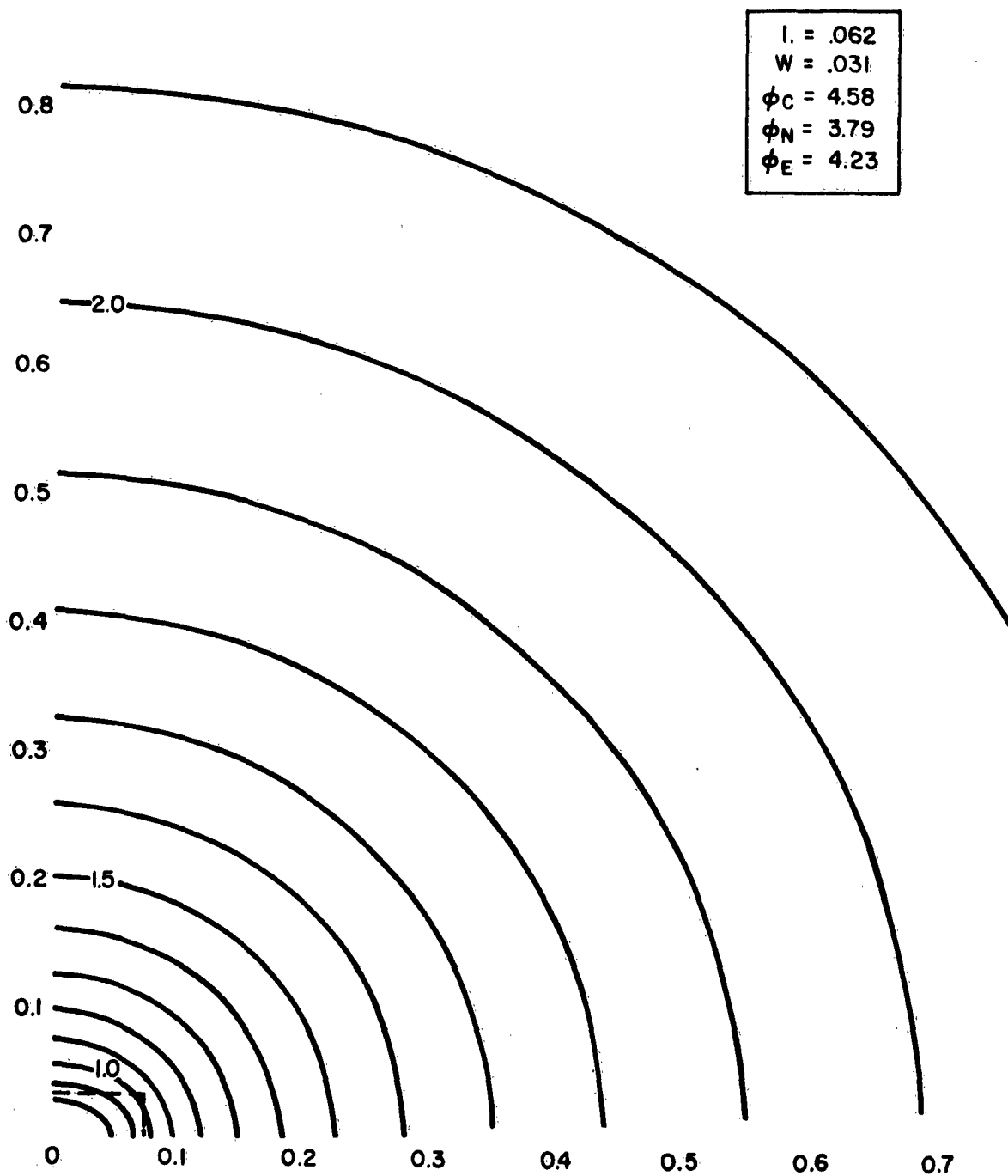


Figure B.10 - Contour Map of Relative Current Density  $\phi/\phi_c$ .

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